Outline of orthonormal basis of wavelets having customizable frequency bands and wide range of wavelet shapes (Wavelet analysis and signal processing)

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1 Introduction

We already proposed the orthonormal wavelet basis with arbitrary real dilation factor[4, 5] and the orthonormal basis of wavelets having customizable frequency bands[6]. Additionally, based on them, we recently proposed a new type of orthonormal basis of wavelets having not only customizable frequency bands, but also a wide range of wavelet shapes in the time domain. In this paper, we introduce its outline (because of space limitations, some proofs of lemmas and theorems are omitted). This basis has flexible scaling functions, which can be translated in the time domain under the control of an arbitrary real constant b, and according to the constant number of b, its wavelets have variable shapes in the time domain.

First, we define the orthonormal basis of wavelets having customizable frequency bands and a wide range of wavelet shapes (Sec. 3), and we prove its orthonormality (Sec. 4). Next, we introduce the perfect translation invariance theorems (PTI theorems),[7, 8] which are useful for designing perfect-translation-invariant wavelet frames[2, 7, 8, 9, 10, 11, 13] and signal quantitative analyses[12] (Sec. 5), and using PTI theorems, we prove that our proposed wavelets construct a basis in $L^2(\mathbb{R})$ (Sec. 6).

2 Preliminaries

$\mathbb{R}$ denotes the set of real numbers, and $\mathbb{Z}$ denotes the set of integers, and $\mathbb{N}$ denotes the set of natural numbers. $L^1(\mathbb{R})$ denotes the space of integrable functions, and $L^2(\mathbb{R})$ denotes the space of square integrable functions. We use the following notation for the inner product of the functions $f(t) \in L^2(\mathbb{R})$ and $g(t) \in L^2(\mathbb{R})$:

$$\langle f\,,\,g\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt. \tag{1}$$

Note that $\overline{g(t)}$ is the complex conjugate of $g(t)$. Next, the norm $\|f\|$ of the function $f(t) \in L^2(\mathbb{R})$ is defined by

$$\|f\| = \sqrt{\langle f\,,\,f\rangle}. \tag{2}$$

The Fourier transform $\hat{f}(\omega)$ of the function $f(t) \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt. \tag{3}$$

The inverse Fourier transform $f(t)$ of the function $\hat{f}(\omega) \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}^{-1}(\hat{f})(t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,e^{i\omega t}\,d\omega. \tag{4}$$

The Kronecker delta $\delta_{k,l}$ is defined by

$$\delta_{k,l} = \begin{cases} 1\,, & k = l, \\ 0\,, & k \neq l, \end{cases} \qquad k, l \in \mathbb{Z}. \tag{5}$$

3 The definition of the orthonormal basis of wavelets $\{\psi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ j, n \in \mathbb{Z}\}$

The positive number sequence $\{R_j : j \in \mathbb{Z}\}$ is defined by

$$0 < \cdots < R_{j-1} < R_j < R_{j+1} < \cdots < \infty, \quad j \in \mathbb{Z}. \tag{6}$$

$$\lim_{j\to-\infty} R_j = 0, \tag{7}$$

$$\lim_{j\to\infty} R_j = \infty. \tag{8}$$

The other positive number sequence $\{\Delta_j : j \in \mathbb{Z}\}$ is defined under the following conditions:

$$\Delta_j > 0, \qquad j \in \mathbb{Z}, \tag{9}$$
$$\Delta_j + \Delta_{j+1} \leq R_{j+1} - R_j, \quad j \in \mathbb{Z}. \tag{10}$$

These sequences $\{R_j : j \in \mathbb{Z}\}$ and $\{\Delta_j : j \in \mathbb{Z}\}$ can be freely designed under the conditions (6)–(10). The bounds of the frequency bands are defined by $\{\pm\pi R_j : j \in \mathbb{Z}\}$, and each bounds $\pm\pi R_j$ have a crossover area whose length is $2\pi\Delta_j$. With an arbitrary real number constant b, each scaling function set $\{\phi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ n \in \mathbb{Z}\}$ of each level j ($j \in \mathbb{Z}$) is defined by

$$\phi^b_{j,n}(t) = \frac{1}{\sqrt{R_j}}\phi_j\left(t - \frac{n+b}{R_j}\right), \qquad n \in \mathbb{Z}, \tag{11}$$

where

$$\hat{\phi}_j(\omega) = \begin{cases} 1\,, & |\omega| \leq \pi(R_j - \Delta_j), \\ \cos\left(\theta_j(|\omega|)\right), & \pi(R_j - \Delta_j) < |\omega| < \pi(R_j + \Delta_j), \\ 0\,, & \text{otherwise}, \end{cases} \tag{12}$$

$$\theta_j(x) = \frac{\pi}{2}\nu\left(\frac{x - \pi(R_j - \Delta_j)}{2\pi\Delta_j}\right), \tag{13}$$

$$\nu(x) = \begin{cases} 0, & x \leq 0, \\ x^4(35 - 84x + 70x^2 - 20x^3), & 0 < x < 1, \\ 1, & x \geq 1. \end{cases} \tag{14}$$

The function $\nu(x)$ in (14) was proposed by Daubechies[1] for Meyer's scaling function,[3] and the following equation holds:

$$\nu(x) + \nu(1-x) = 1. \tag{15}$$

Using $\hat{\phi}_j(\omega)$ in (12), the function set $\{\hat{\psi}^M_j(\omega), \hat{\psi}^P_j(\omega) : j \in \mathbb{Z}\}$ is defined by

$$\hat{\psi}^M_j(\omega) = \begin{cases} \sqrt{\left|\hat{\phi}_{j+1}(\omega)\right|^2 - \left|\hat{\phi}_j(\omega)\right|^2}\,, & \omega < 0, \\ 0\,, & \omega \geq 0, \end{cases} \quad j \in \mathbb{Z}, \tag{16}$$

$$\hat{\psi}^P_j(\omega) = \hat{\psi}^M_j(-\omega), \qquad j \in \mathbb{Z}. \tag{17}$$

The basic wavelet set $\{\Psi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ n \in \mathbb{Z}\}$ of level j $(j \in \mathbb{Z})$ is defined by

$$\Psi^b_{j,n}(t) = \sqrt{p_j}\left[\psi^M_j(t)e^{i\pi\left\{-\beta_j(n+\frac{1}{2})+(b+\frac{1}{2})\right\}} + \psi^P_j(t)e^{i\pi\left\{\beta_j(n+\frac{1}{2})-(b+\frac{1}{2})\right\}}\right], \quad n \in \mathbb{Z}, \tag{18}$$

where

$$p_j = \frac{1}{R_{j+1} - R_j}, \tag{19}$$

$$\beta_j = \frac{R_j}{R_{j+1} - R_j}. \tag{20}$$

The orthonormal basis of wavelets $\{\psi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ j, n \in \mathbb{Z}\}$ is defined by

$$\psi^b_{j,n}(t) = \Psi^b_{j,n}\left(t - p_j\left(n + \frac{1}{2}\right)\right), \quad j,\ n \in \mathbb{Z}. \tag{21}$$

3.1 Some equations associated with $\{\psi^b_{j,n}(t) :$ constant $b \in \mathbb{R},\ j, n \in \mathbb{Z}\}$

With $j \in \mathbb{Z}$, substituting (12) in (16) and (17),

$$\hat{\psi}^M_j(\omega) = \begin{cases} \cos\left(\theta_{j+1}(-\omega)\right), & -\pi(R_{j+1} + \Delta_{j+1}) < \omega < -\pi(R_{j+1} - \Delta_{j+1}), \\ 1, & -\pi(R_{j+1} - \Delta_{j+1}) \le \omega \le -\pi(R_j + \Delta_j), \\ \sin\left(\theta_j(-\omega)\right), & -\pi(R_j + \Delta_j) < \omega < -\pi(R_j - \Delta_j), \\ 0, & \text{otherwise}, \end{cases} \tag{22}$$

$$\hat{\psi}^P_j(\omega) = \hat{\psi}^M_j(-\omega). \tag{23}$$

Lemma 3.1. *With $j \in \mathbb{Z}$, the following equations hold:*

$$\hat{\psi}_j^P(\omega)\overline{\hat{\psi}_{j-1}^P(\omega)} = \hat{\mu}_j(\omega), \tag{24}$$

$$\hat{\psi}_j^M(\omega)\overline{\hat{\psi}_{j-1}^M(\omega)} = \hat{\mu}_j(\omega + 2\pi R_j), \tag{25}$$

$$\hat{\psi}_j^P(\omega)\overline{\hat{\psi}_j^M(\omega - 2\pi R_j)} = \hat{\mu}_j(\omega), \tag{26}$$

$$\hat{\psi}_j^M(\omega)\overline{\hat{\psi}_j^P(\omega + 2\pi R_j)} = \hat{\mu}_j(-\omega), \tag{27}$$

$$\hat{\psi}_j^P(\omega)\overline{\hat{\psi}_j^M(\omega - 2\pi R_{j+1})} = \hat{\mu}_{j+1}(\omega), \tag{28}$$

$$\hat{\psi}_j^M(\omega)\overline{\hat{\psi}_j^P(\omega + 2\pi R_{j+1})} = \hat{\mu}_{j+1}(-\omega), \tag{29}$$

where

$$\hat{\mu}_j(\omega) = \begin{cases} \sin\left(\theta_j\left(\omega\right)\right)\cos\left(\theta_j\left(\omega\right)\right), & \pi(R_j - \Delta_j) < \omega < \pi(R_j + \Delta_j), \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Proof. From (22) and (23), $\hat{\psi}_j^P(\omega)$ is represented as

$$\hat{\psi}_j^P(\omega) = \begin{cases} \sin\left(\theta_j(\omega)\right), & \pi(R_j - \Delta_j) < \omega < \pi(R_j + \Delta_j), \\ 1\,, & \pi(R_j + \Delta_j) \le \omega \le \pi(R_{j+1} - \Delta_{j+1}), \\ \cos\left(\theta_{j+1}(\omega)\right), & \pi(R_{j+1} - \Delta_{j+1}) < \omega < \pi(R_{j+1} + \Delta_{j+1}), \\ 0\,, & \text{otherwise.} \end{cases} \tag{31}$$

Then

$$\begin{aligned} \hat{\psi}_j^P(\omega)\overline{\hat{\psi}_{j-1}^P(\omega)} &= \begin{cases} \sin\left(\theta_j\left(\omega\right)\right)\cos\left(\theta_j\left(\omega\right)\right), & \pi(R_j - \Delta_j) < \omega < \pi(R_j + \Delta_j), \\ 0, & \text{otherwise,} \end{cases} \\ &= \hat{\mu}_j\left(\omega\right). \end{aligned} \tag{32}$$

From (32), we have (24). In the same manner as above, we have (25)–(29). □

4 The proofs of orthonormality

In this section, we prove the orthonormality associated with $\{\psi_{j,n}^b(t) : \text{constant } b \in \mathbb{R},\ j, n \in \mathbb{Z}\}$ and $\{\phi_{j,n}^b(t) : \text{constant } b \in \mathbb{R},\ j, n \in \mathbb{Z}\}$.

Lemma 4.1. *The following equations hold for* $\hat{\psi}_j^M(\omega)$ *and* $\hat{\psi}_j^P(\omega)$:

$$\left|\hat{\psi}_j^M(\omega - \pi R_j)\right|^2 + \left|\hat{\psi}_j^P(\omega + \pi R_j)\right|^2 = 1, \qquad -\pi\Delta_j < \omega < \pi\Delta_j,\ j \in \mathbb{Z}, \tag{33}$$

$$\left|\hat{\psi}_j^M(\omega + \pi R_j - 2\pi R_{j+1})\right|^2 + \left|\hat{\psi}_j^P(\omega + \pi R_j)\right|^2 = 1, \qquad \pi(R_{j+1} - R_j - \Delta_{j+1}) < \omega < \pi(R_{j+1} - R_j + \Delta_{j+1}),\ j \in \mathbb{Z}. \tag{34}$$

Proof. From (22), (23) and (13), $\left|\hat{\psi}_j^M(\omega - \pi R_j)\right|^2 + \left|\hat{\psi}_j^P(\omega + \pi R_j)\right|^2$ within $-\pi\Delta_j < \omega < \pi\Delta_j$ is represented as

$$\begin{aligned}
&\left|\hat{\psi}_j^M(\omega - \pi R_j)\right|^2 + \left|\hat{\psi}_j^P(\omega + \pi R_j)\right|^2 \\
&\quad = \left\{\sin\left(\frac{\pi}{2}\nu\left(\frac{-\omega + \pi\Delta_j}{2\pi\Delta_j}\right)\right)\right\}^2 + \left\{\sin\left(\frac{\pi}{2}\nu\left(\frac{\omega + \pi\Delta_j}{2\pi\Delta_j}\right)\right)\right\}^2 \\
&\quad = \left\{\sin\left(\frac{\pi}{2}\nu\left(1 - y\right)\right)\right\}^2 + \left\{\sin\left(\frac{\pi}{2}\nu\left(y\right)\right)\right\}^2 \\
&\quad = \left\{\cos\left(\frac{\pi}{2}\nu\left(y\right)\right)\right\}^2 + \left\{\sin\left(\frac{\pi}{2}\nu\left(y\right)\right)\right\}^2 \\
&\quad = 1.
\end{aligned} \tag{35}$$

Note that, replacing $\frac{\omega+\pi\Delta_j}{2\pi\Delta_j}$ (within $-\pi\Delta_j < \omega < \pi\Delta_j$) with y (within $0 < y < 1$) in the 2nd line of (35), the 3rd line is derived, and from (15), $\nu(1-y) = 1 - \nu(y)$ and substituting it in the 3rd line of (35), the 4th line is derived. From (35), we have (33), and in the same manner as above, we have (34). □

4.1 The orthonormality of $\{\psi_{j,n}^b(t) : \text{constant } b \in \mathbb{R}, j, n \in \mathbb{Z}\}$

With fixed $b \in \mathbb{R}$ and $j \in \mathbb{Z}$, we prove the orthonormality of $\{\psi_{j,n}^b(t) :$ constant $b \in \mathbb{R},\ n \in \mathbb{Z}\}$ in the same level j by the following Lemma 4.2:

Lemma 4.2. *The following equation holds*:

$$\left\langle \psi_{j,n_1}^b,\ \psi_{j,n_2}^b \right\rangle = \delta_{n_1,n_2}, \qquad b \in \mathbb{R},\ j, n_1, n_2 \in \mathbb{Z}. \tag{36}$$

Proof. $\left\langle \psi^b_{j,n_1},\, \psi^b_{j,n_2} \right\rangle$ is represented as

$$
\begin{aligned}
&\left\langle \psi^b_{j,n_1},\, \psi^b_{j,n_2} \right\rangle \\
&= \frac{1}{2\pi} \left\langle \hat{\psi}^b_{j,n_1},\, \hat{\psi}^b_{j,n_2} \right\rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Psi}^b_{j,n_1}(\omega)\, e^{-ip_j\left(n_1+\frac{1}{2}\right)\omega} \overline{\hat{\Psi}^b_{j,n_2}(\omega)\, e^{-ip_j\left(n_2+\frac{1}{2}\right)\omega}} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Psi}^b_{j,n_1}(\omega)\, \overline{\hat{\Psi}^b_{j,n_2}(\omega)} e^{ip_j(-n_1+n_2)\omega} d\omega \\
&= \frac{p_j}{2\pi} \int_{-\infty}^{\infty} \left\{ \left| \hat{\psi}^M_j(\omega) \right|^2 e^{i\pi\beta_j(-n_1+n_2)} + \left| \hat{\psi}^P_j(\omega) \right|^2 e^{i\pi\beta_j(n_1-n_2)} \right\} e^{ip_j(-n_1+n_2)\omega} d\omega \\
&= \frac{p_j}{2\pi} \left\{ \int_{-\infty}^{\infty} \left| \hat{\psi}^M_j(\omega - \pi R_j) \right|^2 e^{i\{p_j(-n_1+n_2)(\omega-\pi R_j)+\pi\beta_j(-n_1+n_2)\}} d\omega \right. \\
&\quad \left. + \int_{-\infty}^{\infty} \left| \hat{\psi}^P_j(\omega + \pi R_j) \right|^2 e^{i\{p_j(-n_1+n_2)(\omega+\pi R_j)+\pi\beta_j(n_1-n_2)\}} d\omega \right\} \\
&= \frac{1}{2\pi(R_{j+1}-R_j)} \int_{-\infty}^{\infty} \left\{ \left| \hat{\psi}^M_j(\omega - \pi R_j) \right|^2 + \left| \hat{\psi}^P_j(\omega + \pi R_j) \right|^2 \right\} e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega.
\end{aligned}
\tag{37}
$$

Note that, considering (21), the 3rd line of (37) is derived, and substituting (18) in the 4th line of (37) and considering $\hat{\psi}^M_j(\omega)\overline{\hat{\psi}^P_j(\omega)} = \hat{\psi}^P_j(\omega)\overline{\hat{\psi}^M_j(\omega)} = 0$ $(\omega \in \mathbb{R})$ from (22) and (23), the 5th line of (37) is derived, and substituting (19) and (20) in the 6th and 7th line of (37), the 8th line is derived. From (22), (23) and (33) in Lemma 4.1, $|\hat{\psi}^M(\omega - \pi R_j)|^2 + |\hat{\psi}^P(\omega + \pi R_j)|^2$ in (37) is represented as

$$
\left| \hat{\psi}^M_j(\omega - \pi R_j) \right|^2 + \left| \hat{\psi}^P_j(\omega + \pi R_j) \right|^2 =
\begin{cases}
\left| \hat{\psi}^M_j(\omega - \pi R_j) \right|^2, & -A_{R_j} < \omega < -B_{R_j}, \\
1, & -B_{R_j} \le \omega \le B_{R_j}, \\
\left| \hat{\psi}^P_j(\omega + \pi R_j) \right|^2, & B_{R_j} < \omega < A_{R_j}, \\
0, & \text{otherwise},
\end{cases}
\tag{38}
$$

where

$$
A_{R_j} = \pi(R_{j+1} - R_j + \Delta_{j+1}), \tag{39}
$$
$$
B_{R_j} = \pi(R_{j+1} - R_j - \Delta_{j+1}). \tag{40}
$$

Substituting (38) in (37),

$$
\begin{aligned}
&\left\langle \psi^b_{j,n_1},\ \psi^b_{j,n_2}\right\rangle \\
&= \frac{1}{2\pi(R_{j+1}-R_j)}\left[\int_{-A_{R_j}}^{-B_{R_j}} \left|\hat{\psi}^M_j(\omega-\pi R_j)\right|^2 e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega \right.\\
&\quad \left. + \int_{-B_{R_j}}^{B_{R_j}} e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega + \int_{B_{R_j}}^{A_{R_j}} \left|\hat{\psi}^P_j(\omega+\pi R_j)\right|^2 e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega\right] \\
&= \frac{1}{2\pi(R_{j+1}-R_j)} \\
&\quad \times \left[\int_{-A_{R_j}+2\pi(R_{j+1}-R_j)}^{-B_{R_j}+2\pi(R_{j+1}-R_j)} \left|\hat{\psi}^M_j(\omega-2\pi(R_{j+1}-R_j)-\pi R_j)\right|^2 e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega \right.\\
&\quad \left. + \int_{-B_{R_j}}^{B_{R_j}} e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega + \int_{B_{R_j}}^{A_{R_j}} \left|\hat{\psi}^P_j(\omega+\pi R_j)\right|^2 e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega\right] \\
&= \frac{1}{2\pi(R_{j+1}-R_j)}\left[\int_{-B_{R_j}}^{B_{R_j}} e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega \right.\\
&\quad \left. + \int_{B_{R_j}}^{A_{R_j}} \left\{\left|\hat{\psi}^M_j(\omega+\pi R_j-2\pi R_{j+1})\right|^2 + \left|\hat{\psi}^P_j(\omega+\pi R_j)\right|^2\right\} e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega\right] \\
&= \frac{1}{2\pi(R_{j+1}-R_j)}\int_{-B_{R_j}}^{A_{R_j}} e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega \\
&= \frac{1}{2\pi(R_{j+1}-R_j)}\int_{-B_{R_j}}^{-B_{R_j}+2\pi(R_{j+1}-R_j)} e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega} d\omega = \delta_{n_1,n_2}. \qquad (41)
\end{aligned}
$$

Note that, considering the period $2\pi(R_{j+1}-R_j)$ of $e^{i\frac{-n_1+n_2}{R_{j+1}-R_j}\omega}$, the 5th line of (41) is derived, and considering $-A_{R_j}+2\pi(R_{j+1}-R_j) = B_{R_j}$ and $-B_{R_j}+2\pi(R_{j+1}-R_j) = A_{R_j}$ from (39) and (40), the 7th and 8th lines of (41) are derived, and substituting (34) of Lemma 4.1 in 8th line of (41), the 9th line is derived. □

With fixed $b \in \mathbb{R}$ and $j \in \mathbb{Z}$, we prove the orthonormality between $\{\psi^b_{j,n}(t)$: constant $b \in \mathbb{R},\ n \in \mathbb{Z}\}$ of level j and $\{\psi^b_{j-1,n}(t)$: constant $b \in \mathbb{R},\ n \in \mathbb{Z}\}$ of level $j-1$ by the following Lemma 4.3.

Lemma 4.3. *The following equation holds:*

$$\left\langle \psi^b_{j,n_1},\ \psi^b_{j-1,n_2} \right\rangle = 0, \qquad b \in \mathbb{R},\ j, n_1, n_2 \in \mathbb{Z}. \tag{42}$$

Proof. Considering (21), $\left\langle \psi^b_{j,n_1},\ \psi^b_{j-1,n_2} \right\rangle$ is represented as

$$\begin{aligned}\left\langle \psi^b_{j,n_1},\ \psi^b_{j-1,n_2} \right\rangle &= \frac{1}{2\pi} \left\langle \hat{\psi}^b_{j,n_1},\ \hat{\psi}^b_{j-1,n_2} \right\rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Psi}^b_{j,n_1}(\omega)\, \overline{\hat{\Psi}^b_{j-1,n_2}(\omega) e^{iK^p_{j,n_1,n_2}\omega}} d\omega, \end{aligned} \tag{43}$$

where

$$K^p_{j,n_1,n_2} = -p_j\left(n_1 + \frac{1}{2}\right) + p_{j-1}\left(n_2 + \frac{1}{2}\right). \tag{44}$$

$\hat{\Psi}^b_{j,n_1}(\omega)\, \overline{\hat{\Psi}^b_{j-1,n_2}(\omega)}$ in (43) is represented as

$$\begin{aligned} &\hat{\Psi}^b_{j,n_1}(\omega)\, \overline{\hat{\Psi}^b_{j-1,n_2}(\omega)} \\ &= \sqrt{p_j}\left\{ \hat{\psi}^M_j(\omega) e^{i\pi\left\{-\beta_j\left(n_1+\frac{1}{2}\right)+\left(b+\frac{1}{2}\right)\right\}} + \hat{\psi}^P_j(\omega) e^{i\pi\left\{\beta_j\left(n_1+\frac{1}{2}\right)-\left(b+\frac{1}{2}\right)\right\}} \right\} \\ &\times \overline{\sqrt{p_{j-1}}\left\{ \hat{\psi}^M_{j-1}(\omega) e^{i\pi\left\{-\beta_{j-1}\left(n_2+\frac{1}{2}\right)+\left(b+\frac{1}{2}\right)\right\}} + \hat{\psi}^P_{j-1}(\omega) e^{i\pi\left\{\beta_{j-1}\left(n_2+\frac{1}{2}\right)-\left(b+\frac{1}{2}\right)\right\}} \right\}} \\ &= \sqrt{p_j p_{j-1}}\left\{ \hat{\psi}^M_j(\omega)\overline{\hat{\psi}^M_{j-1}(\omega)} e^{i\pi\left\{-\beta_j\left(n_1+\frac{1}{2}\right)+\beta_{j-1}\left(n_2+\frac{1}{2}\right)\right\}} \right. \\ &\quad \left. + \hat{\psi}^P_j(\omega)\overline{\hat{\psi}^P_{j-1}(\omega)} e^{i\pi\left\{\beta_j\left(n_1+\frac{1}{2}\right)-\beta_{j-1}\left(n_2+\frac{1}{2}\right)\right\}} \right\} \\ &= \sqrt{p_j p_{j-1}}\left\{ \hat{\mu}_j(\omega + 2\pi R_j) e^{-iK^\beta_{j,n_1,n_2}} + \hat{\mu}_j(\omega) e^{iK^\beta_{j,n_1,n_2}} \right\}, \end{aligned} \tag{45}$$

where

$$K^\beta_{j,n_1,n_2} = \pi\left\{ \beta_j\left(n_1 + \frac{1}{2}\right) - \beta_{j-1}\left(n_2 + \frac{1}{2}\right) \right\}. \tag{46}$$

Note that, (18) is substituted in the 2nd and 3rd lines of (45), and considering $\hat{\psi}^M_j(\omega)\overline{\hat{\psi}^P_{j-1}(\omega)} = \hat{\psi}^P_j(\omega)\overline{\hat{\psi}^M_{j-1}(\omega)} = 0$ ($\omega \in \mathbb{R}$) from (22) and (23), the 4th and 5th lines of (45) are derived, and (24) and (25) in Lemma 3.1 are substituted in 6th line of (45). Substituting (45) in (43),

$$
\begin{aligned}
&\left\langle \psi^b_{j,n_1},\ \psi^b_{j-1,n_2} \right\rangle \\
&= \frac{\sqrt{p_j p_{j-1}}}{2\pi} \int_{-\infty}^{\infty} \left\{ \hat{\mu}_j(\omega + 2\pi R_j) e^{-iK^{\beta}_{j,n_1,n_2}} + \hat{\mu}_j(\omega) e^{iK^{\beta}_{j,n_1,n_2}} \right\} e^{iK^{p}_{j,n_1,n_2}\omega} d\omega \\
&= \frac{\sqrt{p_j p_{j-1}}}{2\pi} \left\{ \int_{-\infty}^{\infty} \hat{\mu}_j(\omega) e^{i\left\{K^{p}_{j,n_1,n_2}(\omega - 2\pi R_j) - K^{\beta}_{j,n_1,n_2}\right\}} d\omega \right. \\
&\qquad\qquad \left. + \int_{-\infty}^{\infty} \hat{\mu}_j(\omega) e^{i\left(K^{p}_{j,n_1,n_2}\omega + K^{\beta}_{j,n_1,n_2}\right)} d\omega \right\} \\
&= \frac{\sqrt{p_j p_{j-1}}}{2\pi} \int_{-\infty}^{\infty} \hat{\mu}_j(\omega) e^{i\left(K^{p}_{j,n_1,n_2}\omega + K^{\beta}_{j,n_1,n_2}\right)} d\omega \left(e^{-i\left(2\pi R_j K^{p}_{j,n_1,n_2} + 2K^{\beta}_{j,n_1,n_2}\right)} + 1 \right) \\
&= \frac{\sqrt{p_j p_{j-1}}}{2\pi} \int_{-\infty}^{\infty} \hat{\mu}_j(\omega) e^{i\left(K^{p}_{j,n_1,n_2}\omega + K^{\beta}_{j,n_1,n_2}\right)} d\omega \left(e^{-i\pi(2n_2+1)} + 1 \right) \\
&= 0
\end{aligned}
\tag{47}
$$

Note that, from (19), (20), (44) and (46), we have $2\pi R_j K^{p}_{j,n_1,n_2} + 2K^{\beta}_{j,n_1,n_2} = \pi(2n_2 + 1)$ and substituting it in the 5th line of (47), the 6th line is derived. □

The following Theorem 4.4 ensures the orthonormality of $\{\psi^b_{j,n}(t) :$ constant $b \in \mathbb{R},\ j, n \in \mathbb{Z}\}$.

Theorem 4.4. *The following equation holds*:

$$
\left\langle \psi^b_{j_1,n_1},\ \psi^b_{j_2,n_2} \right\rangle = \delta_{j_1,j_2}\delta_{n_1,n_2}, \qquad b \in \mathbb{R},\ j_1, j_2, n_1, n_2 \in \mathbb{Z}. \tag{48}
$$

Proof. When $|j_1 - j_2| > 1$,

$$
\text{supp } \hat{\psi}^b_{j_1,n_1}(\omega)\ \cap\ \text{supp } \hat{\psi}^b_{j_2,n_2}(\omega) = \emptyset, \quad |j_1 - j_2| > 1,\ j_1, j_2, n_1, n_2 \in \mathbb{Z}. \tag{49}
$$

Then we have

$$
\left\langle \psi^b_{j_1,n_1},\ \psi^b_{j_2,n_2} \right\rangle = 0, \qquad |j_1 - j_2| > 1,\ j_1, j_2, n_1, n_2 \in \mathbb{Z}. \tag{50}
$$

From Lemmas 4.2, 4.3 and (50), we have (48). □

4.2 The orthonormality associated with $\{\phi_{j,n}^b(t) :$ constant $b \in \mathbb{R},\ j, n \in \mathbb{Z}\}$

With fixed $b \in \mathbb{R}$ and $j \in \mathbb{Z}$, we prove the orthonormality of $\{\phi_{j,n}^b(t)$: constant $b \in \mathbb{R},\ n \in \mathbb{Z}\}$ in the same level j by the following Theorem 4.5:

Theorem 4.5. *The following equation holds*:

$$\left\langle \phi_{j,n_1}^b,\ \phi_{j,n_2}^b \right\rangle = \delta_{n_1,n_2}, \qquad b \in \mathbb{R},\ j, n_1,\ n_2 \in \mathbb{Z}. \tag{51}$$

Proof. In the same manner as Lemma 4.2, we have (51). □

Next, with fixed $b \in \mathbb{R}$ and $j \in \mathbb{Z}$, we prove the orthonormality between $\{\psi_{j,n}^b(t)$: constant $b \in \mathbb{R}$, $n \in \mathbb{Z}\}$ and $\{\phi_{j,n}^b(t)$: constant $b \in \mathbb{R}$ $n \in \mathbb{Z}\}$ by the following Lemma 4.6.

Lemma 4.6. *The following equation holds*:

$$\left\langle \psi_{j,n_1}^b,\ \phi_{j,n_2}^b \right\rangle = 0, \qquad b \in \mathbb{R},\ j, n_1, n_2 \in \mathbb{Z}. \tag{52}$$

Proof. In the same manner as Lemma 4.3, we have (52). □

From Lemma 4.6, we have the following Theorem 4.7:

Theorem 4.7. *The following equation holds:*

$$\left\langle \psi_{j_1,n_1}^b,\ \phi_{j_2,n_2}^b \right\rangle = 0, \qquad b \in \mathbb{R},\ j_1 \geq j_2,\ j_1, j_2, n_1, n_2 \in \mathbb{Z}. \tag{53}$$

Proof. In the same manner as Theorem 4.4, we have (53). □

5 The PTI theorems

In this section, we introduce the PTI theorems as Theorems 5.4 and 5.5. For these theorems, we need to define some conditions and prove some lemmas.

Condition A. For the Fourier transform of $g(t) \in L^1(\mathbb{R})$, there exist constants $C > 0$ and $\epsilon > 0$ such that

$$|\hat{g}(\omega)| \leq C(1 + |\omega|^2)^{-\frac{1}{2}-\epsilon}. \tag{54}$$

Note that it is obvious that $g(t) \in L^2(\mathbb{R})$ from (54). With a constant $p > 0$, $\{g_n(t) : n \in \mathbb{Z}\}$ is defined by

$$g_n(t) = g(t - pn), \quad n \in \mathbb{Z}. \tag{55}$$

Lemma 5.1. *For any $f(t) \in L^2(\mathbb{R})$ and $\{g_n(t) : n \in \mathbb{Z}\}$ satisfying Condition A, there exists a constant $C_1 > 0$ such that*

$$\sum_{n\in\mathbb{Z}} |\langle f, g_n\rangle|^2 \leq C_1 ||f||^2. \tag{56}$$

Proof. According to Ref. [1], Sec. 3.3.2,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\langle f, g_n\rangle|^2 &= \sum_{n\in\mathbb{Z}} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)} e^{ipn\xi} d\xi \right|^2 \\
&= \frac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}} \left| \int_0^{\frac{2\pi}{p}} \sum_{k\in\mathbb{Z}} \hat{f}\left(\xi + \frac{2\pi}{p}k\right) \overline{\hat{g}\left(\xi + \frac{2\pi}{p}k\right)} e^{ipn\xi} d\xi \right|^2 \qquad (57) \\
&= \frac{1}{2\pi p} \int_0^{\frac{2\pi}{p}} \left| \sum_{k\in\mathbb{Z}} \hat{f}\left(\xi + \frac{2\pi}{p}k\right) \overline{\hat{g}\left(\xi + \frac{2\pi}{p}k\right)} \right|^2 d\xi \qquad (58) \\
&= \frac{1}{2\pi p} \int_0^{\frac{2\pi}{p}} \sum_{k\in\mathbb{Z}} \hat{f}\left(\xi + \frac{2\pi}{p}k\right) \overline{\hat{g}\left(\xi + \frac{2\pi}{p}k\right)} \\
&\quad \times \sum_{l\in\mathbb{Z}} \overline{\hat{f}\left(\xi + \frac{2\pi}{p}l\right)} \hat{g}\left(\xi + \frac{2\pi}{p}l\right) d\xi \qquad (59) \\
&= \frac{1}{2\pi p} \sum_{l\in\mathbb{Z}} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)} \overline{\hat{f}\left(\xi + \frac{2\pi}{p}l\right)} \hat{g}\left(\xi + \frac{2\pi}{p}l\right) d\xi \qquad (60) \\
&= \frac{1}{2\pi p} \int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 |\hat{g}(\xi)|^2 d\xi + \mathrm{Rest}(f) \\
&\leq \frac{1}{p} C^2 ||f||^2 + \mathrm{Rest}(f), \qquad (61)
\end{aligned}$$

where

$$\mathrm{Rest}(f) = \frac{1}{2\pi p} \sum_{l\neq 0} \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)} \overline{\hat{f}\left(\xi + \frac{2\pi}{p}l\right)} \hat{g}\left(\xi + \frac{2\pi}{p}l\right) d\xi. \tag{62}$$

Note that, in (57), considering the period $2\pi/p$ of $e^{\imath pn\xi}$, the integral interval is sliced into pieces of length $2\pi/p$, and using the Plancherel theorem for periodic functions, (58) is derived, and considering that $\sum_{l\in\mathbb{Z}} \overline{\hat{f}\left(\xi+\frac{2\pi}{p}l\right)\hat{g}\left(\xi+\frac{2\pi}{p}l\right)}$ in (59) has the period $2\pi/p$, (60) is derived, and (61) is derived from (54). Next, $|\mathrm{Rest}(f)|$ is represented as

$$
\begin{aligned}
|\mathrm{Rest}(f)| &= \frac{1}{2\pi p}\left|\sum_{l\neq 0}\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,\overline{\hat{f}\left(\xi+\frac{2\pi}{p}l\right)}\hat{g}\left(\xi+\frac{2\pi}{p}l\right)d\xi\right| \\
&\leq \frac{1}{2\pi p}\sum_{l\neq 0}\left\{\int_{-\infty}^{\infty}\left|\hat{f}(\xi)\right|^2\left|\overline{\hat{g}(\xi)}\hat{g}\left(\xi+\frac{2\pi}{p}l\right)\right|d\xi\right\}^{1/2} \\
&\quad\times\left\{\int_{-\infty}^{\infty}\left|\overline{\hat{f}\left(\xi+\frac{2\pi}{p}l\right)}\right|^2\left|\overline{\hat{g}(\xi)}\hat{g}\left(\xi+\frac{2\pi}{p}l\right)\right|d\xi\right\}^{1/2} \\
&= \frac{1}{2\pi p}\sum_{l\neq 0}\left\{\int_{-\infty}^{\infty}\left|\hat{f}(\xi)\right|^2\left|\hat{g}(\xi)\hat{g}\left(\xi+\frac{2\pi}{p}l\right)\right|d\xi\right\}^{1/2} \\
&\quad\times\left\{\int_{-\infty}^{\infty}\left|\hat{f}\left(\eta\right)\right|^2\left|\hat{g}\left(\eta-\frac{2\pi}{p}l\right)\hat{g}(\eta)\right|d\eta\right\}^{1/2} \\
&\leq \frac{1}{2\pi p}\sum_{l\neq 0}\left\{\int_{-\infty}^{\infty}\left|\hat{f}(\xi)\right|^2\beta\left(\frac{2\pi}{p}l\right)d\xi\right\}^{1/2} \\
&\quad\times\left\{\int_{-\infty}^{\infty}\left|\hat{f}\left(\eta\right)\right|^2\beta\left(-\frac{2\pi}{p}l\right)d\eta\right\}^{1/2} \\
&= \frac{1}{p}||f||^2\sum_{l\neq 0}\left\{\beta\left(\frac{2\pi}{p}l\right)\beta\left(-\frac{2\pi}{p}l\right)\right\}^{1/2} \\
&= \frac{2}{p}||f||^2\sum_{l=1}^{\infty}\beta\left(\frac{2\pi}{p}l\right),
\end{aligned}
\tag{63}
$$

where

$$
\beta(s) = \sup_{\xi\in\mathbb{R}}|\hat{g}(\xi)\hat{g}(\xi+s)|. \tag{64}
$$

Note that, using the Cauchy-Schwarz inequality, the 2nd and 3rd lines of (63) are derived, and substituting $\eta = \xi + \frac{2\pi}{p}l$ in the 3rd line of (63), the 4th and

5th lines are derived, and from $\beta(s)=\beta(-s)$, the 9th line of (63) is derived. Next, from (54) and (64), we have

$$\begin{aligned}\beta(s) &\le \sup_{\xi\in\mathbb{R}} C^2(1+|\xi|^2)^{-\frac{1}{2}-\epsilon}(1+|\xi+s|^2)^{-\frac{1}{2}-\epsilon}\\ &\le C_2(1+|s|^2)^{-\frac{1}{2}-\epsilon},\end{aligned} \tag{65}$$

where $C_2>0$ is a constant. Then we have

$$\sum_{l=1}^{\infty}\beta\left(\frac{2\pi}{p}l\right)\le C_2\sum_{l=1}^{\infty}\left(1+\left|\frac{2\pi}{p}l\right|^2\right)^{-\frac{1}{2}-\epsilon}=C_3<\infty, \tag{66}$$

and

$$|\mathrm{Rest}(f)|\le\frac{2}{p}C_3||f||^2. \tag{67}$$

Therefore, with $C_1=\frac{1}{p}C^2+\frac{2}{p}C_3$, we have (56) from (61) and (67). $\square$

Considering (58) in Lemma 5.1, we have the following Corollary 5.2.

Corollary 5.2. *For any $f(t)\in L^2(\mathbb{R})$, $g(t)$ satisfying (54) in Condition A and a constant $p>0$, the following equation holds within $0\le\omega\le 2\pi/p$:*

$$\sum_{k\in\mathbb{Z}}\hat{f}\left(\omega+\frac{2\pi}{p}k\right)\overline{\hat{g}\left(\omega+\frac{2\pi}{p}k\right)}\in L^2\left(0,\frac{2\pi}{p}\right),\qquad 0\le\omega\le\frac{2\pi}{p}, \tag{68}$$

and the left-hand side of (68) has the period $2\pi/p$ in $\omega\in\mathbb{R}$.

Lemma 5.3. *For any $\{a_n:n\in\mathbb{Z}\}\in\ell^2(\mathbb{Z})$ and $\{g_n(t):n\in\mathbb{Z}\}$ satisfying Condition A, the function $G(t)$ is defined by*

$$G(t)=\sum_{n\in\mathbb{Z}}a_n g_n(t). \tag{69}$$

Then we have

$$G(t)\in L^2(\mathbb{R}). \tag{70}$$

Proof. We define $G^N(t)$ $(N \in \mathbb{N})$ by

$$G^N(t) = \sum_{|n|\le N} a_n g_n(t), \quad N \in \mathbb{N}. \tag{71}$$

Then we have

$$\begin{aligned}
\|G^N\| &= \sup_{||h||=1} \left|\langle G^N, h\rangle\right| \\
&= \sup_{||h||=1} \left|\left\langle \sum_{|n|\le N} a_n g_n, h \right\rangle\right| \\
&= \sup_{||h||=1} \left|\sum_{|n|\le N} a_n \langle g_n, h\rangle\right| \\
&\le \left(\sum_{|n|\le N} |a_n|^2\right)^{1/2} \sup_{||h||=1} \left(\sum_{|n|\le N} |\langle h, g_n\rangle|^2\right)^{1/2} \\
&\le \sqrt{C_1} \left(\sum_{|n|\le N} |a_n|^2\right)^{1/2}.
\end{aligned} \tag{72}$$

Note that, using the Cauchy-Schwarz inequality, the 4th line of (72) is derived. Therefore, $\{G^N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R})$, and we have

$$G(t) = \lim_{N\to\infty} G^N(t) \in L^2(\mathbb{R}). \tag{73}$$

□

Theorem 5.4. *We denote by the frame operator* $\mathcal{W}_p^g$, *the transform of* $f(t) \in L^2(\mathbb{R})$ *by* $\{g_n(t) : n \in \mathbb{Z}\}$ *satisfying Condition A*:

$$(\mathcal{W}_p^g f)(t) = \sum_{n\in\mathbb{Z}} \langle f,\ g_n\rangle\, g_n(t). \tag{74}$$

Then we have

$$\mathcal{W}_p^g f \in L^2(\mathbb{R}), \tag{75}$$

$$\mathcal{F}(\mathcal{W}_p^g f)(\omega) = \frac{1}{p}\hat{g}(\omega) \sum_{k\in\mathbb{Z}} \left\{ \overline{\hat{g}\left(\omega - \frac{2\pi}{p}k\right)} \hat{f}\left(\omega - \frac{2\pi}{p}k\right) \right\}. \tag{76}$$

Proof. Since $\{\langle f,\ g_n\rangle : n\in\mathbb{Z}\}\in\ell^2(\mathbb{Z})$ from Lemma 5.1, we have (75) from Lemma 5.3. Therefore, the Fourier transform of $\mathcal{W}_p^g f$ is represented as

$$\begin{aligned}
&\mathcal{F}(\mathcal{W}_p^g f)(\omega)\\
&=\sum_{n\in\mathbb{Z}}\langle f,\ g_n\rangle\,\hat{g}_n(\omega)\\
&=\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}e^{ipn\xi}d\xi\ \hat{g}(\omega)e^{-ipn\omega}\\
&=\frac{1}{p}\hat{g}(\omega)\sum_{n\in\mathbb{Z}}\int_0^{\frac{2\pi}{p}}\sum_{k\in\mathbb{Z}}\hat{f}\left(\xi+\frac{2\pi}{p}k\right)\overline{\hat{g}\left(\xi+\frac{2\pi}{p}k\right)}\sqrt{\frac{p}{2\pi}}e^{ipn\xi}d\xi\ \sqrt{\frac{p}{2\pi}}e^{-ipn\omega}\\
&=\frac{1}{p}\hat{g}(\omega)\sum_{k\in\mathbb{Z}}\left\{\hat{f}\left(\omega+\frac{2\pi}{p}k\right)\overline{\hat{g}\left(\omega+\frac{2\pi}{p}k\right)}\right\}\\
&=\frac{1}{p}\hat{g}(\omega)\sum_{k\in\mathbb{Z}}\left\{\overline{\hat{g}\left(\omega-\frac{2\pi}{p}k\right)}\hat{f}\left(\omega-\frac{2\pi}{p}k\right)\right\}.
\end{aligned}\tag{77}$$

Note that, in the 4th line of (77), considering the period $2\pi/p$ of $e^{ipn\xi}$, the integral interval is sliced into pieces of length $2\pi/p$, and from Corollary 5.2, $\sum_{k\in\mathbb{Z}}\hat{f}\left(\omega+\frac{2\pi}{p}k\right)\overline{\hat{g}\left(\omega+\frac{2\pi}{p}k\right)}$, included in $L^2(0,2\pi/p)$ within $0\leq\omega\leq 2\pi/p$, has the period $2\pi/p$ in $\omega\in\mathbb{R}$, and additionally, $\sum_{n\in\mathbb{Z}}$ or the later is the Fourier series expansion of it in $[0,2\pi/p)$ with the orthogonal basis $\left\{\sqrt{\frac{p}{2\pi}}e^{-ipn\omega}:n\in\mathbb{Z}\right\}$, then the 5th line of (77) is derived, and in the 6th line of (77), $-k$ is replaced with k. □

The next Theorem 5.5 ensures that the transform in $L^2(\mathbb{R})$ by a function set $\{h_n(t)\}_{n\in\mathbb{Z}}$ satisfying the following Condition B has perfect translation invariance:[7, 8]

Condition B. The Fourier transform of $h(t)\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ has a compact support of length Ω_h as follows:

$$0<\sup_{\omega\in\mathbb{R}}|\hat{h}(\omega)|=C_h<\infty,\tag{78}$$

$$\operatorname{supp}\hat{h}(\omega)\subseteq\left[\omega_1^h,\omega_2^h\right]\ ,\ \ \omega_1^h<\omega_2^h,\tag{79}$$

$$0<\Omega_h=\omega_2^h-\omega_1^h<\infty.\tag{80}$$

$\{h_n(t) : n \in \mathbb{Z}\}$ is defined by

$$h_n(t) = h(t - p_h\, n), \quad n \in \mathbb{Z}, \tag{81}$$

where

$$0 < p_h\, \Omega_h \leq 2\pi. \tag{82}$$

Note that $p_h > 0$ is a constant real number satisfying (82).

Theorem 5.5. *We denote by the frame operator $\mathcal{W}^h_{p_h}$, the transform of $f(t) \in L^2(\mathbb{R})$ by $\{h_n(t) : n \in \mathbb{Z}\}$ satisfying Condition B:*

$$(\mathcal{W}^h_{p_h} f)(t) = \sum_{n \in \mathbb{Z}} \langle f,\ h_n \rangle\, h_n(t). \tag{83}$$

Then we have

$$\mathcal{W}^h_{p_h} f \in L^2(\mathbb{R}), \tag{84}$$

$$\mathcal{F}(\mathcal{W}^h_{p_h}\, f)(\omega) = \frac{1}{p_h} |\hat{h}(\omega)|^2 \hat{f}(\omega). \tag{85}$$

Proof. Since $\{h_n(t) : n \in \mathbb{Z}\}$ under the conditions (78)–(82) satisfies Condition A, from Theorem 5.4, we have (84) and

$$\mathcal{F}(\mathcal{W}^h_{p_h}\, f)(\omega) = \frac{1}{p_h} \hat{h}(\omega) \sum_{k \in \mathbb{Z}} \left\{ \overline{\hat{h}\left(\omega - \frac{2\pi}{p_h} k\right)}\, \hat{f}\left(\omega - \frac{2\pi}{p_h} k\right) \right\}. \tag{86}$$

When $\omega \notin \operatorname{supp} \hat{h}(\omega)$, we have $\hat{h}(\omega) = 0$ and $\mathcal{F}(\mathcal{W}^h_{p_h}\, f)(\omega) = 0$, therefore

$$\mathcal{F}(\mathcal{W}^h_{p_h}\, f)(\omega) = 0 = \frac{1}{p_h} \mid \hat{h}(\omega) \mid^2 \hat{f}(\omega), \ \ \omega \notin \operatorname{supp} \hat{h}(\omega). \tag{87}$$

When $\omega \in \operatorname{supp} \hat{h}(\omega)$, as $\Omega_h \leq 2\pi/p_h$ from (82), we have $\hat{h}(\omega - 2\pi k/p_h) = \hat{h}(\omega)$ for $k = 0$, and $\hat{h}(\omega - 2\pi k/p_h) = 0$ for $k \neq 0$. Then (86) is represented as

$$\begin{aligned} \mathcal{F}(\mathcal{W}^h_{p_h}\, f)(\omega) &= \frac{1}{p_h} \hat{h}(\omega) \overline{\hat{h}(\omega)} \hat{f}(\omega) \\ &= \frac{1}{p_h} |\hat{h}(\omega)|^2 \hat{f}(\omega), \quad \omega \in \operatorname{supp} \hat{h}(\omega). \end{aligned} \tag{88}$$

From (87) and (88), we have (85). □

6 The construction of a basis

In this section, we prove that $\{\psi^b_{j,n}(t) : \text{constant } b \in \mathbb{R}\ , j, n \in \mathbb{Z}\}$ construct a basis in $L^2(\mathbb{R})$. First, we need to define some functions. For any $f(t) \in L^2(\mathbb{R})$, $f^b_j(t)$ is defined by

$$f^b_j(t) = \sum_{n\in\mathbb{Z}} \left\langle f,\ \psi^b_{j,n}\right\rangle \psi^b_{j,n}(t), \qquad b \in \mathbb{R},\ j \in \mathbb{Z}. \tag{89}$$

From (18) and (21), $\psi^b_{j,n}(t)$ is represented as

$$\begin{aligned}\psi^b_{j,n}(t) = \psi^M_{j,n}(t)e^{i\pi\left\{-\beta_j(n+\frac{1}{2})+(b+\frac{1}{2})\right\}} + \psi^P_{j,n}(t)e^{i\pi\left\{\beta_j(n+\frac{1}{2})-(b+\frac{1}{2})\right\}},\\ b \in \mathbb{R},\ n, j \in \mathbb{Z}.\end{aligned} \tag{90}$$

where

$$\psi^M_{j,n}(t) = \sqrt{p_j}\,\psi^M_j\left(t - p_j\left(n+\frac{1}{2}\right)\right), \tag{91}$$

$$\psi^P_{j,n}(t) = \sqrt{p_j}\,\psi^P_j\left(t - p_j\left(n+\frac{1}{2}\right)\right). \tag{92}$$

Substituting (90) in (89), $f^b_j(t)$ is represented as

$$\begin{aligned}f^b_j(t) &= \sum_{n\in\mathbb{Z}} \left\langle f,\ \psi^M_{j,n}\right\rangle \psi^M_{j,n}(t) + \sum_{n\in\mathbb{Z}} \left\langle f,\ \psi^P_{j,n}\right\rangle \psi^P_{j,n}(t)\\ &\quad - e^{-i2\pi b}\sum_{n\in\mathbb{Z}} e^{i2\pi\beta_j\left(n+\frac{1}{2}\right)} \left\langle f,\ \psi^M_{j,n}\right\rangle \psi^P_{j,n}(t)\\ &\quad - e^{i2\pi b}\sum_{n\in\mathbb{Z}} e^{-i2\pi\beta_j\left(n+\frac{1}{2}\right)} \left\langle f,\ \psi^P_{j,n}\right\rangle \psi^M_{j,n}(t), \quad b \in \mathbb{R},\ j \in \mathbb{Z}.\end{aligned} \tag{93}$$

Note that, considering $e^{\pm i\pi} = -1$, the right hand of (93) is derived. The functions orig$M^f_j(t)$, orig$P^f_j(t)$, alias$M^f_j(t)$, alias$P^f_j(t)$ are defined by

$$\mathrm{orig}M^f_j(t) = \sum_{n\in\mathbb{Z}} \left\langle f,\ \psi^M_{j,n}\right\rangle \psi^M_{j,n}(t), \qquad j \in \mathbb{Z}, \tag{94}$$

$$\mathrm{orig}P^f_j(t) = \sum_{n\in\mathbb{Z}} \left\langle f,\ \psi^P_{j,n}\right\rangle \psi^P_{j,n}(t), \qquad j \in \mathbb{Z}, \tag{95}$$

$$\mathrm{alias}M^f_j(t) = \sum_{n\in\mathbb{Z}} e^{i2\pi\beta_j\left(n+\frac{1}{2}\right)} \left\langle f,\ \psi^M_{j,n}\right\rangle \psi^P_{j,n}(t), \quad j \in \mathbb{Z}, \tag{96}$$

$$\mathrm{alias}P^f_j(t) = \sum_{n\in\mathbb{Z}} e^{-i2\pi\beta_j\left(n+\frac{1}{2}\right)} \left\langle f,\ \psi^P_{j,n}\right\rangle \psi^M_{j,n}(t), \quad j \in \mathbb{Z}. \tag{97}$$

From (93)–(97),

$$f_j^b(t) = \mathrm{orig}M_j^f(t) + \mathrm{orig}P_j^f(t) - e^{-i2\pi b}\mathrm{alias}M_j^f(t) - e^{i2\pi b}\mathrm{alias}P_j^f(t),$$
$$b \in \mathbb{R},\ j \in \mathbb{Z}. \quad (98)$$

6.1 The proof of $\sum_{j\in\mathbb{Z}} \left\{\mathrm{orig}M_j^f(t) + \mathrm{orig}P_j^f(t)\right\} = f(t)$

Lemma 6.1. *The following equations hold:*

$$\mathcal{F}\left(\mathrm{orig}M_j^f\right)(\omega) = \left|\hat{\psi}_j^M(\omega)\right|^2 \hat{f}(\omega), \quad j \in \mathbb{Z}, \quad (99)$$
$$\mathcal{F}\left(\mathrm{orig}P_j^f\right)(\omega) = \left|\hat{\psi}_j^P(\omega)\right|^2 \hat{f}(\omega), \quad j \in \mathbb{Z}. \quad (100)$$

Proof. From (19) and (91), $\psi_{j,n}^M(t)$ is represented as

$$\psi_{j,n}^M(t) = \psi_{j,0}^M(t - p_j n), \quad (101)$$

where

$$p_j = \frac{1}{R_{j+1} - R_j}. \quad (102)$$

From (91), the Fourier transform of $\psi_{j,0}^M(t)$ is

$$\hat{\psi}_{j,0}^M(\omega) = \sqrt{p_j}\,\hat{\psi}_j^M(\omega)\, e^{-i\frac{1}{2}p_j\omega}. \quad (103)$$

From (22) and (103), the compact support length Ω_j^M of $\hat{\psi}_{j,0}^M(\omega)$ is represented as

$$\Omega_j^M = \omega_2 - \omega_1 = \pi\left(R_{j+1} - R_j + \Delta_j + \Delta_{j+1}\right), \quad (104)$$

where

$$\mathrm{supp}\ \hat{\psi}_{j,0}^M(\omega) = [\omega_1,\ \omega_2], \quad (105)$$
$$\omega_1 = -\pi\left(R_{j+1} + \Delta_{j+1}\right), \quad (106)$$
$$\omega_2 = -\pi\left(R_j - \Delta_j\right). \quad (107)$$

From (102) and (104),

$$p_j\Omega_j^M = \pi\left(1+\frac{\Delta_j+\Delta_{j+1}}{R_{j+1}-R_j}\right)$$
$$\leq 2\pi. \tag{108}$$

Note that, considering (10), the 2nd line of (108) is derived. Since $\{\psi_{j,n}^M(t) : n \in \mathbb{Z}\}$ satisfies Condition B (that is, (78)–(82) hold for $h(t) = \psi_{j,0}^M(t)$, $\Omega_h = \Omega_j^M$ and $p_h = p_j$ with fixed $j \in \mathbb{Z}$), from Theorem 5.5 and (94),

$$\mathcal{F}\left(\mathrm{orig}M_j^f\right)(\omega) = \frac{1}{p_j}\left|\hat{\psi}_{j,0}^M(\omega)\right|^2 \hat{f}(\omega)$$
$$= \left|\hat{\psi}_j^M(\omega)\right|^2 \hat{f}(\omega). \tag{109}$$

Note that (103) is substituted in the 2nd line of (109). From (109), we have (99), and in the same manner as above, we have (100). □

Theorem 6.2. *The following equation holds*:

$$\sum_{j\in\mathbb{Z}}\left\{\mathrm{orig}M_j^f(t) + \mathrm{orig}P_j^f(t)\right\} = f(t). \tag{110}$$

Proof. From (99) and (100) in Lemma 6.1,

$$\sum_{j\in\mathbb{Z}}\left\{\mathcal{F}\left(\mathrm{orig}M_j^f\right)(\omega) + \mathcal{F}\left(\mathrm{orig}P_j^f\right)(\omega)\right\}$$
$$= \left\{\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j^M(\omega)\right|^2 + \sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j^P(\omega)\right|^2\right\}\hat{f}(\omega). \tag{111}$$

Considering (22), $\sum_{j=j_1}^{j_2}\left|\hat{\psi}_j^M(\omega)\right|^2$ $(j_1 < j_2,\ j_1, j_2 \in \mathbb{Z})$ is represented as

$$\sum_{j=j_1}^{j_2}\left|\hat{\psi}_j^M(\omega)\right|^2 = \begin{cases} \left|\cos\left(\theta_{j_2+1}(-\omega)\right)\right|^2, \\ \qquad -\pi(R_{j_2+1}+\Delta_{j_2+1}) < \omega < -\pi(R_{j_2+1}-\Delta_{j_2+1}), \\ 1, \qquad -\pi(R_{j_2+1}-\Delta_{j_2+1}) \leq \omega \leq -\pi(R_{j_1}+\Delta_{j_1}), \\ \left|\sin\left(\theta_{j_1}(-\omega)\right)\right|^2, \\ \qquad -\pi(R_{j_1}+\Delta_{j_1}) < \omega < -\pi(R_{j_1}-\Delta_{j_1}), \\ 0, \qquad \text{otherwise}, \end{cases} \tag{112}$$

Therefore, considering (6)–(10),

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j^M(\omega)\right|^2 = \lim_{j_1\to-\infty}\lim_{j_2\to\infty}\sum_{j=j_1}^{j_2}\left|\hat{\psi}_j^M(\omega)\right|^2 = \begin{cases} 0\,, & \omega\geq 0, \\ 1\,, & \omega<0. \end{cases} \tag{113}$$

In the same manner as above,

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j^P(\omega)\right|^2 = \begin{cases} 1\,, & \omega>0, \\ 0\,, & \omega\leq 0. \end{cases} \tag{114}$$

From (111), (113) and (114), we have (110). □

6.2 The proof of $\sum_{j\in\mathbb{Z}} \mathrm{alias}M_j^f(t) = 0$ and $\sum_{j\in\mathbb{Z}} \mathrm{alias}P_j^f(t) = 0$

Lemma 6.3. *The following equations hold:*

$$\mathcal{F}\left(\mathrm{alias}M_j^f\right)(\omega) = \hat{\psi}_j^P(\omega)\Big\{\overline{\hat{\psi}_j^M(\omega-2\pi R_j)}\hat{f}(\omega-2\pi R_j) - \overline{\hat{\psi}_j^M(\omega-2\pi R_{j+1})}\hat{f}(\omega-2\pi R_{j+1})\Big\}, \tag{115}$$

$$\mathcal{F}\left(\mathrm{alias}P_j^f\right)(\omega) = \hat{\psi}_j^M(\omega)\Big\{\overline{\hat{\psi}_j^P(\omega+2\pi R_j)}\hat{f}(\omega+2\pi R_j) - \overline{\hat{\psi}_j^P(\omega+2R_{j+1})}\hat{f}(\omega+2\pi R_{j+1})\Big\}. \tag{116}$$

Proof. From (22) and (91), $\{\psi_{j,n}^M(t) : n\in\mathbb{Z}\}$ satisfies Condition A with fixed $j\in\mathbb{Z}$ (that is, (54) holds for $\hat{g}(\omega)=\hat{\psi}_{j,0}^M(\omega)$ with fixed $j\in\mathbb{Z}$). Therefore, from Lemma 5.1, for any $f(t)\in L^2(\mathbb{R})$, there exists a constant $C_4>0$ such that

$$\sum_{n\in\mathbb{Z}} |\langle f,\psi_{j,n}^M\rangle|^2 \leq C_4||f||^2. \tag{117}$$

Then we have

$$\sum_{n\in\mathbb{Z}} \left| e^{i2\pi\beta_j\left(n+\frac{1}{2}\right)} \left\langle f, \psi_{j,n}^{M}\right\rangle \right|^2 = \sum_{n\in\mathbb{Z}} \left|\left\langle f, \psi_{j,n}^{M}\right\rangle\right|^2 \le C_4\|f\|^2, \tag{118}$$

and $\left\{e^{i2\pi\beta_j\left(n+\frac{1}{2}\right)}\left\langle f,\ \psi_{j,n}^{M}\right\rangle : n\in\mathbb{Z}\right\} \in \ell^2(\mathbb{Z})$. Additionally, from (22), (23) and (92), $\left\{\psi_{j,n}^{P}(t) : n\in\mathbb{Z}\right\}$ satisfies Condition A, then we have alias$M_j^f(t) \in L^2(\mathbb{R})$ from Lemma 5.3. Therefore, from (91) and (92), the Fourier transform of alias$M_j^f(t)$ in (96) is represented as

$$\begin{aligned}\mathcal{F}\left(\mathrm{alias}M_j^f\right)(\omega) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\left\{e^{i2\pi\beta_j\left(n+\frac{1}{2}\right)}\left\langle \hat{f},\ \hat{\psi}_{j,n}^{M}\right\rangle \hat{\psi}_{j,n}^{P}(\omega)\right\} \\ &= \frac{p_j}{2\pi}\sum_{n\in\mathbb{Z}}\left\{e^{i2\pi\beta_j\left(n+\frac{1}{2}\right)}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{\psi}_j^{M}(\xi)}e^{ip_j\left(n+\frac{1}{2}\right)\xi}d\xi\right. \\ &\qquad \left.\times\ \hat{\psi}_j^{P}(\omega)\, e^{-ip_j\left(n+\frac{1}{2}\right)\omega}\right\}.\end{aligned} \tag{119}$$

Now, we define $\mathcal{F}\left(\mathrm{alias}M_{j,m}^f\right)(\omega)$ $(m\in\mathbb{N})$ by

$$\begin{aligned}\mathcal{F}\left(\mathrm{alias}M_{j,m}^f\right)(\omega) &= \frac{p_j}{2\pi}\sum_{n\in\mathbb{Z}}\left\{e^{i2\pi\frac{[m\beta_j]}{m}\left(n+\frac{1}{2}\right)}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{\psi}_j^{M}(\xi)}e^{ip_j\left(n+\frac{1}{2}\right)\xi}d\xi\right. \\ &\qquad \left.\times\ \hat{\psi}_j^{P}(\omega)\, e^{-ip_j\left(n+\frac{1}{2}\right)\omega}\right\}, \qquad m\in\mathbb{N},\end{aligned} \tag{120}$$

where

$$[x] = \max\{n\in\mathbb{Z} : n\le x\},\ \ x\in\mathbb{R}. \tag{121}$$

Next, replacing n with $m\times n + l$ $(0\le l < m,\ n, l\in\mathbb{Z})$, $\mathcal{F}\left(\mathrm{alias}M_{j,m}^f\right)(\omega)$

in (120) is represented as

$$
\begin{aligned}
&\mathcal{F}\left(\mathrm{alias}M^f_{j,m}\right)(\omega)\\
&=\frac{p_j}{2\pi}\sum_{l=0}^{m-1}\sum_{n\in\mathbb{Z}}\left\{e^{i2\pi\frac{[m\beta_j]}{m}\left(mn+l+\frac{1}{2}\right)}\int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{\psi}_j^M(\xi)}e^{ip_j\left(mn+l+\frac{1}{2}\right)\xi}d\xi\right.\\
&\quad\left.\times\ \hat{\psi}_j^P(\omega)\,e^{-ip_j\left(mn+l+\frac{1}{2}\right)\omega}\right\}\\
&=\frac{p_j}{2\pi}e^{i\pi\frac{[m\beta_j]}{m}}\hat{\psi}_j^P(\omega)\,e^{-i\frac{p_j}{2}\omega}\sum_{l=0}^{m-1}e^{-ilp_j\omega}e^{i2\pi\frac{[m\beta_j]}{m}l}\\
&\quad\times\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty}\hat{f}(\xi)\,\overline{\hat{\psi}_j^M(\xi)}e^{ip_j\left(l+\frac{1}{2}\right)\xi}e^{ip_jmn\xi}d\xi e^{-ip_jmn\omega}\\
&=\frac{e^{i\pi\frac{[m\beta_j]}{m}}}{m}\hat{\psi}_j^P(\omega)\,e^{-i\frac{p_j}{2}\omega}\sum_{l=0}^{m-1}e^{-ilp_j\omega}e^{i2\pi\frac{[m\beta_j]}{m}l}\\
&\quad\times\sum_{n\in\mathbb{Z}}\int_0^{\frac{2\pi}{p_jm}}\sum_{k\in\mathbb{Z}}\hat{f}\left(\xi+\frac{2\pi}{p_jm}k\right)\overline{\hat{\psi}_j^M\left(\xi+\frac{2\pi}{p_jm}k\right)}\\
&\quad\times e^{ip_j\left(l+\frac{1}{2}\right)\left(\xi+\frac{2\pi}{p_jm}k\right)}\sqrt{\frac{p_jm}{2\pi}}e^{ip_jmn\xi}d\xi\sqrt{\frac{p_jm}{2\pi}}e^{-ip_jmn\omega}\\
&=\frac{e^{i\pi\frac{[m\beta_j]}{m}}}{m}\hat{\psi}_j^P(\omega)\,e^{-i\frac{p_j}{2}\omega}\sum_{l=0}^{m-1}e^{-ilp_j\omega}e^{i2\pi\frac{[m\beta_j]}{m}l}\\
&\quad\times\sum_{k\in\mathbb{Z}}\hat{f}\left(\omega+\frac{2\pi}{p_jm}k\right)\overline{\hat{\psi}_j^M\left(\omega+\frac{2\pi}{p_jm}k\right)}e^{ip_j\left(l+\frac{1}{2}\right)\left(\omega+\frac{2\pi}{p_jm}k\right)}\\
&=\frac{e^{i\pi\frac{[m\beta_j]}{m}}}{m}\hat{\psi}_j^P(\omega)\sum_{l=0}^{m-1}\sum_{k\in\mathbb{Z}}\hat{f}\left(\omega+\frac{2\pi}{p_jm}k\right)\overline{\hat{\psi}_j^M\left(\omega+\frac{2\pi}{p_jm}k\right)}e^{i\left(2\pi\frac{[m\beta_j]+k}{m}l+\pi\frac{k}{m}\right)}.
\end{aligned}
\tag{122}
$$

Note that, considering $e^{i2\pi[m\beta_j]n}=1$, the 4th and 5th lines of (122) are derived, and in the 6th–8th lines of (122), considering the period $\frac{2\pi}{p_jm}$ of $e^{ip_jmn\xi}$, the integral interval is sliced into pieces of the length $\frac{2\pi}{p_jm}$, and since (54) in Condition A holds for $\hat{g}(\omega)=\hat{\psi}_j^M(\omega)e^{-ip_j\left(l+\frac{1}{2}\right)\omega}$ with fixed j, from Corollary 5.2, $\sum_{k\in\mathbb{Z}}\hat{f}\left(\omega+\frac{2\pi}{p_jm}k\right)\overline{\hat{\psi}_j^M\left(\omega+\frac{2\pi}{p_jm}k\right)}e^{ip_j\left(l+\frac{1}{2}\right)\left(\omega+\frac{2\pi}{p_jm}k\right)}$, included in

$L^2(0, \frac{2\pi}{p_j m})$ within $0 \leq \omega \leq \frac{2\pi}{p_j m}$, has the period $\frac{2\pi}{p_j m}$ in $\omega \in \mathbb{R}$, and additionally, $\sum_{n\in\mathbb{Z}}$ or the later is the Fourier series expansion of it in $[0, \frac{2\pi}{p_j m})$ with the orthogonal basis $\left\{\sqrt{\frac{p_j m}{2\pi}} e^{-ip_j mn\omega} : n \in \mathbb{Z}\right\}$, then the 9th and 10th lines of (122) are derived. Considering the compact supports of $\hat{\psi}_j^M(\omega)$ and $\hat{\psi}_j^P(\omega)$ in (22) and (23), $\mathcal{F}\left(\text{alias}M_{j,m}^f\right)(\omega)$ in (122) can be represented as a sum of a finite number of compact supported functions, and $\frac{e^{i\pi\frac{[m\beta_j]}{m}}}{m}\sum_{l=0}^{m-1} e^{i\left(2\pi\frac{[m\beta_j]+k}{m}l+\pi\frac{k}{m}\right)}$ in (122) is represented as

$$\frac{e^{i\pi\frac{[m\beta_j]}{m}}}{m}\sum_{l=0}^{m-1} e^{i\left(2\pi\frac{[m\beta_j]+k}{m}l+\pi\frac{k}{m}\right)} = \begin{cases} (-1)^{k_2}, & k = -[m\beta_j] - k_2 m,\ k_2 \in \mathbb{Z}, \\ 0\ , & \text{otherwise.} \end{cases} \tag{123}$$

Note that, when $k = -[m\beta_j] - k_2 m,\ k_2 \in \mathbb{Z}$, we have $\frac{[m\beta_j]+k}{m} \in \mathbb{Z}$ and $e^{i2\pi\frac{[m\beta_j]+k}{m}l} = 1$, and when $k \neq -[m\beta_j] - k_2 m,\ \forall k_2 \in \mathbb{Z}$, we have $\frac{[m\beta_j]+k}{m} \notin \mathbb{Z}$ and considering $[m\beta_j] + k \in \mathbb{Z}$ and $m \in \mathbb{N}$, we have $\sum_{l=0}^{m-1} e^{i2\pi\frac{[m\beta_j]+k}{m}l} = 0$, therefore (123) is derived. From (122) and (123),

$$\begin{aligned}\mathcal{F}\left(\text{alias}M_{j,m}^f\right)(\omega) = \hat{\psi}_j^P(\omega)\sum_{k_2\in\mathbb{Z}}\Bigg\{(-1)^{k_2}\hat{f}\left(\omega - \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right)\right) \\ \times \overline{\hat{\psi}_j^M\left(\omega - \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right)\right)}\Bigg\}.\end{aligned} \tag{124}$$

We consider the compact supports of $\hat{\psi}_j^P(\omega)$ and $\hat{\psi}_j^M\left(\omega - \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right)\right)$ in the following two cases:

1. The case of $k_2 \leq -1$: From (22) and (23), the compact supports of $\hat{\psi}_j^P(\omega)$ and $\hat{\psi}_j^M(\omega)$ are represented as

$$\text{supp}\ \hat{\psi}_j^M(\omega) = \left[-A_{S_j}, -B_{S_j}\right], \tag{125}$$

$$\text{supp}\ \hat{\psi}_j^P(\omega) = \left[B_{S_j}, A_{S_j}\right], \tag{126}$$

where

$$A_{S_j} = \pi(R_{j+1} + \Delta_{j+1}), \tag{127}$$

$$B_{S_j} = \pi(R_j - \Delta_j), \tag{128}$$

and the compact support of $\hat{\psi}^M\left(\omega - \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right)\right)$ is represented as

$$\begin{aligned}
&\operatorname{supp} \hat{\psi}^M\left(\omega - \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right)\right) \\
&= \left[-A_{S_j} + \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right), -B_{S_j} + \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right)\right].
\end{aligned} \tag{129}$$

From (128),

$$\begin{aligned}
&B_{S_j} - \left\{-B_{S_j} + \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right)\right\} \\
&\geq 2B_{S_j} - \frac{2\pi}{p_j}(\beta_j - 1) \\
&= 2\pi(R_{j+1} - R_j - \Delta_j) \\
&> 0.
\end{aligned} \tag{130}$$

Note that, considering $\frac{[m\beta_j]}{m} \leq \beta_j$ and $k_2 \leq -1$, the 2nd line of (130) is derived, and (19), (20) and (128) are substituted in the 3rd line of (130), and the 4nd line of (130) is derived from (9) and (10). Then, we have

$$B_{S_j} > -B_{S_j} + \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right). \tag{131}$$

From (126), (129) and (131),

$$\operatorname{supp} \hat{\psi}_j^P(\omega) \cap \operatorname{supp} \hat{\psi}_j^M\left(\omega - \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right)\right) = \emptyset, \quad k_2 \leq -1. \tag{132}$$

2. The case of $k_2 \geq 2$: We consider the condition for $\frac{[m\beta_j]}{m}$, in which the following inequality holds for $k_2 \geq 2$:

$$-A_{S_j} + \frac{2\pi}{p_j}\left(\frac{[m\beta_j]}{m} + k_2\right) > A_{S_j} \tag{133}$$

If (133) holds for $k_2 = 2$, it holds for any $k_2 \geq 2$ ($k_2 \in \mathbb{Z}$). Then, substituting $k_2 = 2$ in (133), the condition of (133) is represented as

$$\begin{aligned}\frac{[m\beta_j]}{m} &> \frac{p_j}{\pi}A_{S_j} - 2 \\ &= \frac{-R_{j+1} + 2R_j + \Delta_{j+1}}{R_{j+1} - R_j}.\end{aligned} \tag{134}$$

Note that (19) and (127) are substituted in the 2nd line of (134). From (20),

$$\begin{aligned}\beta_j - \frac{-R_{j+1} + 2R_j + \Delta_{j+1}}{R_{j+1} - R_j} &= \frac{R_{j+1} - R_j - \Delta_{j+1}}{R_{j+1} - R_j} \\ &> 0.\end{aligned} \tag{135}$$

Note that, from (9) and (10), the 2nd line of (135) is derived. Then we have

$$\frac{-R_{j+1} + 2R_j + \Delta_{j+1}}{R_{j+1} - R_j} < \beta_j. \tag{136}$$

Considering $\frac{[m\beta_j]}{m} \leq \beta_j$ and from (134) and (136), the condition for $\frac{[m\beta_j]}{m}$ is represented as

$$\frac{-R_{j+1} + 2R_j + \Delta_{j+1}}{R_{j+1} - R_j} < \frac{[m\beta_j]}{m} \leq \beta_j. \tag{137}$$

Considering $\lim_{m\to\infty}[m\beta_j]/m = \beta_j$, there exists $M \in \mathbb{N}$ such that, for any $m' > M$ ($m' \in \mathbb{N}$), the following inequality holds:

$$\frac{-R_{j+1} + 2R_j + \Delta_{j+1}}{R_{j+1} - R_j} < \frac{[m'\beta_j]}{m'} \leq \beta_j, \quad m' > M. \tag{138}$$

For $m = m'$, (133) holds, and from (126) and (129),

$$\begin{aligned}\operatorname{supp}\hat{\psi}^P(\omega) \cap \operatorname{supp}\hat{\psi}^M\left(\omega - \frac{2\pi}{p_j}\left(\frac{[m'\beta_j]}{m'} + k_2\right)\right) &= \emptyset, \\ m' > M,\ k_2 \geq 2.\end{aligned} \tag{139}$$

Considering the above two cases, from (124), (132) and (139),

$$\begin{aligned}
&\mathcal{F}\left(\text{alias}M^f_{j,m'}\right)(\omega)\\
&=\hat{\psi}^P(\omega)\sum_{k_2=0}^{1}\left\{(-1)^{k_2}\hat{f}\left(\omega-\frac{2\pi}{p_j}\left(\frac{[m'\beta_j]}{m'}+k_2\right)\right)\right.\\
&\quad\left.\times\overline{\hat{\psi}^M\left(\omega-\frac{2\pi}{p_j}\left(\frac{[m'\beta_j]}{m'}+k_2\right)\right)}\right\}\\
&=\hat{\psi}^P(\omega)\left\{\overline{\hat{\psi}^M\left(\omega-\frac{2\pi}{p_j}\frac{[m'\beta_j]}{m'}\right)}\hat{f}\left(\omega-\frac{2\pi}{p_j}\frac{[m'\beta_j]}{m'}\right)\right.\\
&\quad\left.-\overline{\hat{\psi}^M\left(\omega-\frac{2\pi}{p_j}\left(\frac{[m'\beta_j]}{m'}+1\right)\right)}\hat{f}\left(\omega-\frac{2\pi}{p_j}\left(\frac{[m'\beta_j]}{m'}+1\right)\right)\right\},\quad m'>M.
\end{aligned}\tag{140}$$

Considering $\lim_{m'\to\infty}[m'\beta_j]/m'=\beta_j$, from (119), (120) and (140),

$$\begin{aligned}
&\mathcal{F}\left(\text{alias}M^f_{j}\right)(\omega)\\
&=\lim_{m'\to\infty}\mathcal{F}\left(\text{alias}M^f_{j,m'}\right)(\omega)\\
&=\hat{\psi}^P_j(\omega)\left\{\overline{\hat{\psi}^M\left(\omega-\frac{2\pi}{p_j}\beta_j\right)}\hat{f}\left(\omega-\frac{2\pi}{p_j}\beta_j\right)\right.\\
&\quad\left.-\overline{\hat{\psi}^M\left(\omega-\frac{2\pi}{p_j}(\beta_j+1)\right)}\hat{f}\left(\omega-\frac{2\pi}{p_j}(\beta_j+1)\right)\right\}\\
&=\hat{\psi}^P_j(\omega)\left\{\overline{\hat{\psi}^M_j(\omega-2\pi R_j)}\hat{f}(\omega-2\pi R_j)-\overline{\hat{\psi}^M_j(\omega-2\pi R_{j+1})}\hat{f}(\omega-2\pi R_{j+1})\right\}.
\end{aligned}\tag{141}$$

Note that (19) and (20) are substituted in the 5th line of (141). Then we have (115), and in the same manner as above, we have (116). □

Theorem 6.4. *The following equations hold*:

$$\sum_{j\in\mathbb{Z}}\text{alias}M^f_j(t)=0,\tag{142}$$

$$\sum_{j\in\mathbb{Z}}\text{alias}P^f_j(t)=0.\tag{143}$$

Proof. Substituting (26) and (28) in (115) of Lemma 6.3,

$$\begin{aligned}\mathcal{F}\left(\text{alias}M_j^f\right)(\omega) &= \hat{\mu}_j(\omega)\hat{f}(\omega - 2\pi R_j) - \hat{\mu}_{j+1}(\omega)\hat{f}(\omega - 2\pi R_{j+1}). \\ &= \lambda_j(\omega) - \lambda_{j+1}(\omega), \end{aligned} \tag{144}$$

where

$$\lambda_j(\omega) = \hat{\mu}_j(\omega)\hat{f}(\omega - 2\pi R_j), \qquad j \in \mathbb{Z}. \tag{145}$$

From (30) and (145), $\lambda_j(\omega)$ has the following compact support in $\omega > 0$:

$$\text{supp } \lambda_j(\omega) \subseteq \left[\pi(R_j - \Delta_j),\ \pi(R_j + \Delta_j)\right]. \tag{146}$$

Therefore,

$$\sum_{j\in\mathbb{Z}} \mathcal{F}\left(\text{alias}M_j^f(\omega)\right) = 0. \tag{147}$$

Note that, from (144), $\{\lambda_j(\omega) : j \in \mathbb{Z}\}$ are canceled each other to zero in (147). In the same manner as above,

$$\sum_{j\in\mathbb{Z}} \mathcal{F}\left(\text{alias}P_j^f(\omega)\right) = 0. \tag{148}$$

□

6.3 The proof of $f(t) = \sum_{j,n\in\mathbb{Z}} \left\langle f,\ \psi_{j,n}^b\right\rangle \psi_{j,n}^b(t)$

Theorem 6.5. *For any $f(t) \in L^2(\mathbb{R})$, we have*

$$f(t) = \sum_{j,n\in\mathbb{Z}} \left\langle f,\ \psi_{j,n}^b\right\rangle \psi_{j,n}^b(t), \quad b \in \mathbb{R}. \tag{149}$$

Proof. From Theorems 6.2 and 6.4,

$$f(t) = \sum_{j\in\mathbb{Z}} \left\{\text{orig}M_j^f(t) + \text{orig}P_j^f(t) - e^{-i2\pi b}\text{alias}M_j^f(t) - e^{i2\pi b}\text{alias}P_j^f(t)\right\}. \tag{150}$$

From (89) and (98), $\sum_{j,n\in\mathbb{Z}} \left\langle f,\ \psi^b_{j,n}\right\rangle \psi^b_{j,n}(t)$ is represented as

$$\begin{aligned}
&\sum_{j,n\in\mathbb{Z}} \left\langle f,\ \psi^b_{j,n}\right\rangle \psi^b_{j,n}(t) \\
&\quad = \sum_{j\in\mathbb{Z}} f^b_j(t) \\
&\quad = \sum_{j\in\mathbb{Z}} \left\{ \mathrm{orig}M^f_j(t) + \mathrm{orig}P^f_j(t) - e^{-i2\pi b}\mathrm{alias}M^f_j(t) - e^{i2\pi b}\mathrm{alias}P^f_j(t) \right\}.
\end{aligned} \tag{151}$$

From (150) and (151), we have (149). □

From Theorems 4.4 and 6.5, $\{\psi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ j,n \in \mathbb{Z}\}$ construct an orthonormal basis of wavelets.

6.4 The theorem associated with $\{\phi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ j,n \in \mathbb{Z}\}$

Lemma 6.6. *We denote by the operator $\mathcal{S}^{\phi^b_j}$, the transform of $f(t) \in L^2(\mathbb{R})$ by the scaling function set $\{\phi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ n \in \mathbb{Z}\}$ with fixed $b \in \mathbb{R}$ and $j \in \mathbb{Z}$:*

$$(\mathcal{S}^{\phi^b_j} f)(t) = \sum_{n\in\mathbb{Z}} \left\langle f,\ \phi^b_{j,n}\right\rangle \phi^b_{j,n}(t), \quad b \in \mathbb{R},\ j \in \mathbb{Z}. \tag{152}$$

Then we have

$$\begin{aligned}
\mathcal{F}(\mathcal{S}^{\phi^b_j} f)(\omega) &= \left|\hat{\phi}_j(\omega)\right|^2 \hat{f}(\omega) \\
&\quad + e^{-i2\pi b}\hat{\mu}_j(\omega)\hat{f}(\omega - 2\pi R_j) + e^{i2\pi b}\hat{\mu}_j(-\omega)\hat{f}(\omega + 2\pi R_j).
\end{aligned} \tag{153}$$

Note that $\hat{\mu}_j(\omega)$ is defined by (30) *in Lemma 3.1.*

Proof. Considering Theorem 5.4, we have (153). □

Lemma 6.7. *The following equation holds for $f_j^b(t)$ in* (89):

$$\hat{f}_j^b(\omega) = \left\{ \left| \hat{\psi}_j^M(\omega) \right|^2 + \left| \hat{\psi}_j^P(\omega) \right|^2 \right\} \hat{f}(\omega)$$
$$+ e^{-\imath 2\pi b} \left\{ -\hat{\mu}_j(\omega) \hat{f}(\omega - 2\pi R_j) + \hat{\mu}_{j+1}(\omega) \hat{f}(\omega - 2\pi R_{j+1}) \right\}$$
$$+ e^{\imath 2\pi b} \left\{ -\hat{\mu}_j(-\omega) \hat{f}(\omega + 2\pi R_j) + \hat{\mu}_{j+1}(-\omega) \hat{f}(\omega + 2\pi R_{j+1}) \right\},$$
$$b \in \mathbb{R},\ j \in \mathbb{Z}. \qquad (154)$$

Note that $\hat{\mu}_j(\omega)$ is defined by (30) *in Lemma 3.1.*

Proof. Considering (99), (100) in Lemma 6.1 and (115), (116) in Lemma 6.3, we have (154) from (98). □

Theorem 6.8. *The following equation folds for any $f(t) \in L^2(\mathbb{R})$:*

$$\sum_{n \in \mathbb{Z}} \left\langle f,\ \phi_{j+1,n}^b \right\rangle \phi_{j+1,n}^b(t) = \sum_{n \in \mathbb{Z}} \left\langle f,\ \phi_{j,n}^b \right\rangle \phi_{j,n}^b(t) + \sum_{n \in \mathbb{Z}} \left\langle f,\ \psi_{j,n}^b \right\rangle \psi_{j,n}^b(t),$$
$$b \in \mathbb{R},\ j \in \mathbb{Z}. \qquad (155)$$

Proof. From (153) in Lemma 6.6 and (154) in Lemma 6.7, we have (155). □

7 Conclusions

In this paper, we proposed a new type of orthonormal basis of wavelets having customizable frequency bands and a wide range of wavelet shapes. The main results can be summarized as follows:

1. We defined the orthonormal basis of wavelets $\{\psi_{j,n}^b(t) : \text{constant } b \in \mathbb{R},\ j,n \in \mathbb{Z}\}$ with the scaling functions $\{\phi_{j,n}^b(t) : \text{constant } b \in \mathbb{R},\ j,n \in \mathbb{Z}\}$.

2. We proved the orthonormality of $\{\psi_{j,n}^b(t), \phi_{j,n}^b(t) : \text{constant } b \in \mathbb{R},\ j,n \in \mathbb{Z}\}$.

3. We introduced the PTI theorems, which are useful for designing PTI wavelet frames and signal quantitative analyses.

4. Using the PTI theorems, we proved that $\{\psi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ j,n \in \mathbb{Z}\}$ constructs a basis in $L^2(\mathbb{R})$. Additionally, we also introduced the theorem representing the relations of the spaces spanned by $\{\psi^b_{j,n}(t),$ $\phi^b_{j,n}(t) : \text{constant } b \in \mathbb{R},\ j,n \in \mathbb{Z}\}$.

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